

METRIC APPROXIMATIONS OF WREATH PRODUCTS

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ABSTRACT. Given the large class of groups already known to be sofic, there is seemingly a shortfall in results concerning their permanence properties. We address this problem for wreath products, and in particular investigate the behaviour of more general metric approximations of groups under wreath products.

Our main result is the following. Suppose that H is a sofic group and G is a countable, discrete group. If G is sofic, hyperlinear, weakly sofic, or linear sofic, then $G \wr H$ is also sofic, hyperlinear, weakly sofic, or linear sofic respectively. In each case we construct relevant metric approximations, extending a general construction of metric approximations for $G \wr H$ that uses soficity of H .

1. INTRODUCTION

Sofic groups, introduced by Gromov [13] and developed by Weiss [25], are a large class of groups that can be approximated, in some sense, by finite groups. We consider sofic groups, as well as several other classes of groups which can be similarly defined by metric approximations, namely weakly sofic groups (introduced by Glebsky and Rivera [12]), linear sofic groups (introduced by Arzhantseva and Paunescu [1]) and hyperlinear groups (implicitly defined by Connes and explicitly by Rădulescu [5, 22]).

Via their approximations, sofic, hyperlinear, linear sofic, and weakly sofic groups have applications to a wide area of fields. For example, sofic groups are relevant to ergodic theory [2, 16], topological dynamics, in particular Gottschalk's surjunctivity conjecture [13, 16], group rings and Kaplansky's direct finiteness conjecture [8] (also for linear sofic groups [1, Prop 2.6]), and L^2 -invariants [9, 17]. Hyperlinear groups are of interest in operator algebras, particularly the Connes embedding theorem [5], and in group theory, particularly for the Kevare conjecture [20, Cor 10.4]. We refer the reader to [20, 3] for surveys on sofic and hyperlinear groups.

There are many examples of sofic groups, including all amenable groups, all residually finite groups, and all linear groups (by Malcev's Theorem). However, because of the weakness of the approximation by finite groups, few permanence properties of soficity are properly understood. Relatively straight-forward examples include closure under direct product and increasing unions, and the soficity of residually sofic groups. More substantial results generally require some amenability assumption. For example, an amalgamated product of two sofic groups is known to be sofic if the amalgamated subgroup is amenable (see [11, 19, 6, 21]). This was extended to encompass the fundamental groups of all graphs of groups with sofic vertex groups and amenable edge groups [4]. In the same paper, it is shown that the graph product of sofic groups is sofic. Also, if H is a subgroup of G that is sofic and coamenable, then G is sofic too [10].

Our first result is a new permanence property for soficity.

Theorem 1. *Let G, H be countable, discrete, sofic groups. Then $G \wr H$ is sofic.*

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When G is abelian, Theorem 1 was proved by Paunescu [19], who used methods of analysis and the notion of sofic equivalence relations developed by Elek and Lippner [7]. Holt and Rees recently also proved soficity of $G \wr H$ when H is residually finite and G sofic [14].

We remark that, using the Magnus embedding (see [23] for both the original and a modern geometric definition), Theorem 1 implies the following (in fact it follows from the weaker version of Paunescu, mentioned above [19]).

Corollary 2. *Let N be a normal subgroup of a finite rank free group F , and let N' be the derived subgroup of N . If F/N is sofic, then F/N' is sofic.*

Proof. The Magnus embedding is $F/N' \hookrightarrow \mathbb{Z}^r \wr F/N$. Since soficity passes to subgroups we therefore get the corollary from Theorem 1, or [19]. \square

Sofic groups are also weakly sofic, linear sofic and hyperlinear, and we ask to what extent these properties are preserved by wreath products. Weakly sofic groups are a class of groups which can be approximated by finite groups in a weaker sense than sofic groups, namely one is allowed to approximate G by any finite group with any bi-invariant metric, instead of just permutation groups with the Hamming distance, as is the case for soficity (see Section 4 for precise definitions). Linear soficity and hyperlinearity are each classes of groups which can be approximated by linear groups—linear soficity requires approximation by general linear groups with respect to the rank metric, while hyperlinearity requires approximation by unitary groups in the normalized Hilbert-Schmidt distance.

Our techniques proving Theorem 1 generalize to give the following, broader result.

Theorem 3. *Let G, H be countable, discrete groups and assume that H is sofic. Then:*

- (i) *If G is sofic, then so is $G \wr H$,*
- (ii) *If G is hyperlinear, then so is $G \wr H$,*
- (iii) *If G is linear sofic, then so is $G \wr H$,*
- (iv) *If G is weakly sofic, then so is $G \wr H$.*

The proof of Theorem 3 is constructive, and is almost entirely self-contained, the exceptions being the use of equivalent definitions of soficity, hyperlinearity, and linear soficity, and a result used for (iii) that concerns the behaviour of Jordan blocks under tensor products. The first step in the proof of Theorem 3 is a general result on metric approximations of groups, the proof of which is quantitative (see Proposition 3.1).

We remark that the arguments in the matricial cases (ii),(iii) are more delicate, as each of these arguments require tensor products of operators. In (iii), for example, linear soficity of G allows us to find almost homomorphisms $\theta: G \rightarrow \mathrm{GL}_n(\mathbb{F})$, for some field \mathbb{F} , so that $\frac{1}{n} \mathrm{Rank}(\theta(g) - \mathrm{Id})$ is bounded away from zero for $g \in G \setminus \{1\}$. However, this property is not stable under taking tensor products: for example if $\frac{1}{n} \mathrm{Rank}(\theta(g) - 5 \mathrm{Id})$ and $\frac{1}{n} \mathrm{Rank}(\theta(h) - \frac{1}{5} \mathrm{Id})$ are both small for some $g, h \in G$, then $\frac{1}{n^2} \mathrm{Rank}(\theta(g) \otimes \theta(h) - \mathrm{Id})$ will be small. Because of this issue, we have to remark that linear soficity in fact implies that we can find an almost homomorphism $\theta: G \rightarrow \mathrm{GL}_n(\mathbb{F})$ so that $\inf_{\lambda \in \mathbb{F} \setminus \{0\}} \frac{1}{n} \mathrm{Rank}(\theta(g) - \lambda \mathrm{Id})$ is bounded away from zero for $g \in G \setminus \{1\}$. A similar issue occurs in the hyperlinear case, where we find an almost homomorphism $\theta: G \rightarrow \mathcal{U}(n)$ so that $\theta(g)$ stays a bounded distance away from the scalar matrices (a result of Radulescu [22] enables us to do this for case (ii)). In each of these cases, forcing the image of our group elements to be far away from the scalars is a property that is stable under tensor products. This is a direct computation in the unitary case, whereas the argument that this is true in the general linear case is more involved (see Proposition 4.7).

The structure of the paper is as follows. Section 2 contains the definition of \mathcal{C} -approximable groups, and the definition of sofic groups that we use. This section also looks at how we may

determine that a map from a wreath product to a group is almost multiplicative, and how we endow our wreath products with suitable metrics. Once this is established, we give the initial construction of the metric approximations of a wreath product in Section 3, before extending this to each of the specific cases of Theorem 3 in Section 4.

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2. PRELIMINARIES

We begin with the necessary definitions, as well as a useful lemma to help us identify metric approximations in wreath products.

Throughout we will use 1 to denote the identity element of a group (we expect the reader to be able to infer which group it comes from), except when we talk of the identity matrix, when we use Id.

The *metric approximations*, to which we have referred, can be defined as an embedding of a group into a metric ultraproduct of groups, each with a given bi-invariant metric. Such an embedding gives rise to a sequence of maps to the groups in the ultraproduct. It is these maps on which we focus our attention. We remind the reader that a metric d on a group H is said to be bi-invariant if $d(axb, ayb) = d(x, y)$ for all $a, b, x, y \in H$.

Definition 1. Let H be a group with a bi-invariant metric d . Fix a group G and a function $\theta: G \rightarrow H$.

(a) Given $F \subseteq G$ and $\varepsilon > 0$ we say that θ is (F, ε) -multiplicative if $\theta(1) = 1$ and

$$\max_{g, h \in F} d(\theta(gh), \theta(g)\theta(h)) < \varepsilon.$$

(b) Given $F \subseteq G$ and a function $c: G \setminus \{1\} \rightarrow (0, \infty)$ we say that θ is (F, c) -injective if

$$\min_{g \in F \setminus \{1\}} d(\theta(g), 1) \geq c(g).$$

We remark that we will use the phrases *almost multiplicative* and *almost injective* to mean (F, ε) -multiplicative and (F, c) -injective respectively when we do not wish to specify F, ε and c .

Definition 2. Let \mathcal{C} be a class of pairs (G, d) , where G is a group and d a bi-invariant metric on G (the same group may appear multiple times in \mathcal{C} with different metrics). We say that a group G is \mathcal{C} -approximable if there is a function $c: G \setminus \{1\} \rightarrow (0, \infty)$ so that for every finite $F \subseteq G$ and $\varepsilon > 0$ there is a pair $(H, d) \in \mathcal{C}$ and an (F, ε) -multiplicative function $\theta: G \rightarrow H$ which is also (F, c) -injective.

A special example of \mathcal{C} -approximable groups are sofic groups, where \mathcal{C} consists of the finite symmetric groups paired with the normalized Hamming distance (see [9]).

Definition 3. Let A be a finite set. The *normalized Hamming distance*, denoted $d_{\text{Ham}}(\pi, \tau)$, on $\text{Sym}(A)$ is defined by

$$d_{\text{Ham}}(\pi, \tau) = \frac{1}{|A|} |\{a \in A : \pi(a) \neq \tau(a)\}|.$$

For our purposes, we will use an alternative (but equivalent) definition of soficity (see [9, Thm. 1]).

Definition 4. Let G be a countable discrete group, F a finite subset of G , and $\varepsilon > 0$. Fix a finite set A and a function $\sigma: G \rightarrow \text{Sym}(A)$. We say that σ is (F, ε) -free if

$$\min_{g \in F \setminus \{1\}} d_{\text{Hamm}}(\sigma(g), 1) > 1 - \varepsilon.$$

We say that σ is an (F, ε) -sofic approximation if it is (F, ε) -multiplicative, and (F, ε) -free. Lastly, we say that G is sofic if for every finite $F \subseteq G$ and $\varepsilon > 0$, there is a finite set F and an (F, ε) -sofic approximation $\sigma: G \rightarrow \text{Sym}(A)$.

Our aim is to start with approximations for G and H and use them to build approximations for the wreath product $G \wr H$. We will use generalized wreath products in our approximations, and we recall here the definitions. Note that all our wreath products are of the restricted variety, meaning we use direct sums rather than direct products.

Definition 5. Let X be a set on which H acts. The *generalized wreath product* is defined as

$$G \wr_X H = \bigoplus_X G \rtimes H$$

where the action of $h \in H$ is given via $\alpha_h \in \text{Aut}(\bigoplus_X G)$, defined by a coordinate shift:

$$\alpha_h((g_x)_{x \in X}) = (g_{h^{-1}x})_{x \in X}.$$

The regular wreath product $G \wr H$ is defined as above, taking $X = H$ with H acting on itself by left-multiplication.

A homomorphism $\varphi: G \wr H \rightarrow K$, for some group K , can be decomposed into a pair of homomorphisms $\varphi_1: \bigoplus_H G \rightarrow K$, $\varphi_2: H \rightarrow K$ which satisfy the following equivariance condition:

$$\varphi_2(h)\varphi_1(g) = \varphi_1(\alpha_h(g))\varphi_2(h), \text{ for all } h \in H, g \in \bigoplus_H G.$$

The following lemma gives an analogue to this for the case of almost multiplicative maps.

Lemma 2.1. *Let G, H be countable, discrete groups. For every finite set $F_0 \subseteq G \wr H$ there are finite sets $E_1 \subseteq \bigoplus_H G$, $E_2 \subseteq H$ such that the following holds: Let $\varepsilon > 0$ and K be a group with a bi-invariant metric d . Suppose $\Theta: G \wr H \rightarrow K$ is a map with $\Theta(1) = 1$ such that*

- *the restriction of Θ to $\bigoplus_H G$ is $(E_1, \varepsilon/6)$ -multiplicative,*
- *the restriction of Θ to H is $(E_2, \varepsilon/6)$ -multiplicative,*
- $\max_{g \in E_1, h \in E_2} d(\Theta(g, h), \Theta(g, 1)\Theta(1, h)) < \varepsilon/6,$
- $\max_{g \in E_1, h \in E_2} d(\Theta(1, h)\Theta(g, 1), \Theta(\alpha_h(g), 1)\Theta(1, h)) < \varepsilon/6.$

Then Θ is (F_0, ε) -multiplicative.

Proof. In the following we let $\text{proj}_H: G \wr H \rightarrow H$ and $\text{proj}_G: G \wr H \rightarrow \bigoplus_H G$ be the natural projection maps (note that the latter is not a homomorphism).

After making the right definitions for E_1, E_2 , we apply the triangle inequality several times to obtain the result. We require that if $(g, h), (g', h')$ are in F_0 , then $g, g', \alpha_h(g') \in E_1$, and $h, h' \in E_2$. This is true if we define

$$\begin{aligned} E_1 &= \{\alpha_h(g) : h \in \text{proj}_H(F_0) \cup \{1\}, g \in \text{proj}_G(F_0)\}, \\ E_2 &= \text{proj}_H(F_0). \end{aligned}$$

We leave verification that this is sufficient to the reader. □

Given a group K with a bi-invariant metric d and a finite set B , we will need to construct an appropriate bi-invariant metric on $K \wr_B \text{Sym}(B)$. For this, we use an equivalent description of bi-invariant metrics.

Definition 6. Let G be a group. A function $\ell: G \rightarrow [0, \infty)$ is a *length function* if

- $\ell(g) = \ell(g^{-1})$ for all $g \in G$,
- $\ell(gh) \leq \ell(g) + \ell(h)$ for all $g \in G$.

We say that ℓ is *conjugacy invariant* if $\ell(xgx^{-1}) = \ell(g)$ for all $x, g \in G$.

A conjugacy invariant length function ℓ on G defines a bi-invariant metric by

$$d(x, y) = \ell(y^{-1}x).$$

Conversely if G has a bi-invariant metric d , then $\ell(x) = d(x, 1)$ is a conjugacy invariant length function.

The following proposition establishes how we will endow the wreath product in our approximations with a conjugacy invariant length function. We remark that this proposition is also shown in [14, §5], but we will give a proof for completeness.

Proposition 2.2. Let L be a group with a conjugacy invariant length function ℓ and suppose that $\ell(g) \leq 1$ for all $g \in G$. For a finite set B , define $\tilde{\ell}$ on $L \wr_B \text{Sym}(B)$ by

$$\tilde{\ell}((k_b)_{b \in B}, \tau) = d_{\text{Hamm}}(\tau, 1) + \frac{1}{|B|} \sum_{\substack{b \in B \\ \tau(b)=b}} \ell(k_b).$$

Then $\tilde{\ell}$ is a conjugacy-invariant length function.

Proof. We first show that $\tilde{\ell}$ is conjugacy invariant. Fix $h = (h_b), k = (k_b) \in \bigoplus_B L$ and $\pi, \tau \in \text{Sym}(B)$. Then

$$(k, \tau)^{-1}(h, \pi)(k, \tau) = (\alpha_{\tau^{-1}}(k^{-1}h)\alpha_{\tau^{-1}\pi}(k), \tau^{-1}\pi\tau).$$

Using the bi-invariance of d_{Hamm} we have:

$$\tilde{\ell}((k, \tau)^{-1}(h, \pi)(k, \tau)) = d_{\text{Hamm}}(\pi, 1) + \frac{1}{|B|} \sum_{\substack{b \in B \\ \tau^{-1}\pi\tau(b)=b}} \ell(k_{\tau(b)}^{-1}h_{\tau(b)}k_{\pi^{-1}\tau(b)}).$$

Note that if $\tau^{-1}\pi\tau(b) = b$, then $\tau(b) = \pi^{-1}\tau(b)$. We can use this to rewrite the summation term above, and then use the conjugacy invariance of ℓ to further simplify it:

$$\frac{1}{|B|} \sum_{\substack{b \in B \\ \pi\tau(b)=\tau(b)}} \ell(k_{\tau(b)}^{-1}h_{\tau(b)}k_{\tau(b)}) = \frac{1}{|B|} \sum_{\substack{b \in B \\ \pi\tau(b)=\tau(b)}} \ell(h_{\tau(b)}) = \frac{1}{|B|} \sum_{\substack{b \in B \\ \pi(b)=b}} \ell(h_b).$$

Thus we see that

$$\tilde{\ell}((k, \tau)^{-1}(h, \pi)(k, \tau)) = d_{\text{Hamm}}(\pi, 1) + \frac{1}{|B|} \sum_{\substack{b \in B \\ \pi(b)=b}} \ell(h_b) = \tilde{\ell}(h, \pi).$$

The proof that $\tilde{\ell}((k, \pi)^{-1}) = \tilde{\ell}(k, \pi)$ is similar.

We now prove the triangle inequality. Take h, k, π, τ as above. Then

$$\begin{aligned} \ell((k, \tau)(h, \pi)) &= d_{\text{Hamm}}(\tau\pi, 1) + \frac{1}{|B|} \sum_{\substack{b \in B \\ \pi(b) = \tau^{-1}(b)}} \ell(k_b h_{\tau^{-1}(b)}) \\ &= d_{\text{Hamm}}(\tau\pi, 1) + \frac{1}{|B|} \sum_{\substack{b \in B \\ \pi(b) = \tau^{-1}(b)}} \ell(k_b h_{\pi(b)}) \end{aligned}$$

Let $\hat{B} = \{b \in B : \pi(b) = \tau^{-1}(b) \neq b\}$. Then, using the fact that ℓ is bounded by 1 we get

$$\ell((k, \tau)(h, \pi)) \leq d_{\text{Hamm}}(\tau\pi, 1) + \frac{1}{|B|} \left(|\hat{B}| + \sum_{\substack{b \in B \\ \tau(b) = b}} \ell(k_b) + \sum_{\substack{b \in B \\ \pi(b) = b}} \ell(h_b) \right).$$

From the above we see that it is enough to show that

$$d_{\text{Hamm}}(\tau\pi, 1) + \frac{|\hat{B}|}{|B|} \leq d_{\text{Hamm}}(\tau, 1) + d_{\text{Hamm}}(\pi, 1).$$

Using the definition of the Hamming Distance, we need

$$|\{b : \pi(b) \neq \tau^{-1}(b)\}| + |\hat{B}| \leq |\{b \in B : \tau(b) \neq b\}| + |\{b \in B : \pi(b) \neq b\}|.$$

Since $\hat{B} \subseteq \{b \in B : \pi(b) = b\}$, we get that the above is the same as:

$$|\{b : \pi(b) \neq \tau^{-1}(b)\}| \leq |\{b \in B : \tau^{-1}(b) \neq b\}| + |\{b \in B : \pi(b) \neq b, \pi(b) \neq \tau^{-1}(b)\}|,$$

which we can deduce from the inclusion

$$\{b : \pi(b) \neq \tau^{-1}(b)\} \subseteq \{b \in B : \tau^{-1}(b) \neq b\} \cup \{b \in B : \pi(b) \neq b, \pi(b) \neq \tau^{-1}(b)\}.$$

This completes the proof of the triangle inequality and thus of Proposition 2.2. \square

3. CONSTRUCTION OF THE APPROXIMATION

In the following we let G, H, K be groups, B a finite set, and we suppose that functions $\theta : G \rightarrow K$ and $\sigma : H \rightarrow \text{Sym}(B)$ (not necessarily homomorphisms) are given.

3.1. Some intuition. To give some idea of the intuition behind the construction that follows consider first how one can think of an element of a wreath product $G \wr H$. One may consider $(g, h) \in G \wr H$, where $g = (g_x)_{x \in H}$, as a journey through H , starting at the identity, finishing at h , and picking up elements of G at selected points of H (namely, pick up g_x at x whenever $g_x \neq 1$).

If both G, H are sofic, we wish to construct a finite model for $G \wr H$ using symmetric groups (here $K = \text{Sym}(A)$). A sofic approximation, roughly speaking, gives us a finite set (A or B), inside of which a significant part of the set behaves like a prescribed finite subset of G or H respectively (see e.g. [13, p. 157], [8, Prop 4.4]). We ultimately seek such a set for $G \wr H$, and first we may try to combine A and B in a way which mimics the wreath product of groups. However this approach leads to a problem.

The problem is that in the approximation of H using B there is no prescribed point in B representing the identity. Thus the “journey” through H from the identity to h will translate to a “journey” in B from β to $b = \sigma(h)\beta$, where the choice of β is arbitrary, and may be allowed to vary.

It is for this reason that, in the construction below for sofic groups, we ultimately use $\bigoplus_B A \oplus B$ as our finite set (see Section 4.1). This could be interpreted as using one copy of A for each choice of “identity vertex” in B .

In the initial construction of Section 3.2, we use the ideas above to first obtain a map to $\bigoplus_B \text{Sym}(A) \wr_B \text{Sym}(B)$ (or more generally $\bigoplus_B K \wr_B \text{Sym}(B)$), which is later extended to give a map to $\text{Sym}(\bigoplus_B A \oplus B)$.

3.2. The construction. The aim of this section is to define an approximation of $G \wr H$ into a wreath product that is, in some sense, smaller, or more controllable, than $G \wr H$.

Given the maps $\theta: G \rightarrow K$ and $\sigma: H \rightarrow \text{Sym}(B)$ we define

$$\Theta: G \wr H \rightarrow \bigoplus_B K \wr_B \text{Sym}(B).$$

For $h \in H, b \in B$, define

$$\theta_b^{(h)}: G \rightarrow \bigoplus_B K$$

by $\theta_b^{(h)}(g) = (k_\beta)_{\beta \in B}$ where

$$k_\beta = \begin{cases} \theta(g), & \text{if } \beta = \sigma(h)^{-1}b, \\ 1, & \text{otherwise.} \end{cases}$$

Now suppose that $E \subseteq H$ is finite, and use it to define a subset B_E of B as follows. Set

$$B_1 = \{b \in B : \sigma(h_1)^{-1}b \neq \sigma(h_2)^{-1}b \text{ for all } h_1, h_2 \in E, h_1 \neq h_2\},$$

$$B_2 = \{b \in B : \sigma(h_1 h_2)^{-1}b = \sigma(h_2)^{-1} \sigma(h_1)^{-1}b \text{ for all } h_1, h_2 \in E\},$$

$$B_E = B_1 \cap B_2.$$

Note that $\theta_b^{(h_1)}(g_1)$ and $\theta_b^{(h_2)}(g_2)$ commute if $b \in B_1, h_1, h_2 \in E, g_1, g_2 \in G$ and $h_1 \neq h_2$. Thus it makes sense to define, for $b \in B_E$,

$$\theta_b: \bigoplus_E G \rightarrow \bigoplus_B K$$

by

$$\theta_b((g_h)_{h \in E}) = \prod_{h \in E} \theta_b^{(h)}(g_h).$$

In our applications σ will be a sofic approximation, so we can think of B_E as making up the majority of B . Thus θ_b will be defined for “most” $b \in B$. We extend θ_b to be defined for all $b \in B_E$ by saying that θ_b maps everything to the identity for $b \in B \setminus B_E$. We package all these maps together as a single map

$$\theta_B: \bigoplus_E G \rightarrow \bigoplus_B \left(\bigoplus_B K \right)$$

by

$$\theta_B(g) = (\theta_b(g))_{b \in B}$$

and extend θ_B to $\bigoplus_H G$ by declaring that $\theta_B(g) = 1$ if $g \in \bigoplus_H G$, but $g \notin \bigoplus_E G$.

An equivalent way to view the function θ_B is as follows. If we consider an element of $\bigoplus_B(\bigoplus_B K)$ as a function $B \times B \rightarrow K$, then θ_B is the map defined by

$$(g_h)_{h \in H} \mapsto \begin{cases} (b, \beta) \mapsto \theta(g_h), & \text{for } h \text{ so that } b = \sigma(h)\beta, & \text{if } g_h = 1 \text{ for all } h \notin E \text{ and } b \in B_E, \\ (b, \beta) \mapsto 1, & \text{otherwise.} \end{cases}$$

Referring back to the intuition given in Section 3.1, we see if we fix β , the second coordinate in $B \times B$, and furthermore $\beta \in \bigcap_{h \in E} \sigma(h)^{-1} B_E$, then θ_B restricts to a map that can be expressed as $(g_h)_{h \in E} \mapsto (\theta(g_h))_{\sigma(h)\beta \in B}$. Thus, we can think of β as behaving as the chosen “identity vertex” in B . If an element of $G \wr H$ is a “journey” through H , picking up elements of G en route, then for each such β , the map θ_B tells you where to “pick up” the elements of K on the image of this journey through B , if it starts at β , and provided the original “journey” through H stays in the finite set E .

We now consider

$$\left(\bigoplus_B K \right) \wr_B \text{Sym}(B) = \bigoplus_B \left(\bigoplus_B K \right) \rtimes \text{Sym}(B),$$

so $\pi \in \text{Sym}(B)$ acts on $\bigoplus_B (\bigoplus_B K)$ by α_π where

$$\alpha_\pi((k_b)_{b \in B}) = (k_{\pi^{-1}(b)})_{b \in B}, \text{ if } k_b \in \bigoplus_B K \text{ for all } b \in B.$$

Note that if we identify $k \in \bigoplus_B (\bigoplus_B K)$ with an element $(k_{b,b'})_{b,b' \in B}$ in $\bigoplus_{B \oplus B} K$, then

$$\alpha_\pi((k_{b,b'})_{b,b' \in B}) = (k_{\pi^{-1}(b),b'})_{b,b' \in B}.$$

We define

$$\Theta: G \wr H \rightarrow \left(\bigoplus_B K \right) \wr_B \text{Sym}(B)$$

by

$$\Theta(g, h) = (\theta_B(g), \sigma(h)).$$

We will prove that if K has a bi-invariant metric and θ and σ are approximately multiplicative and sufficiently injective, then Θ gives us our desired approximately multiplicative and sufficiently injective map. To do this, we need to produce an appropriate bi-invariant length function on $(\bigoplus_B K) \wr_B \text{Sym}(B)$.

Let ℓ' be a conjugacy invariant length function on $\bigoplus_B K$ and take $\tilde{\ell}$ to be the conjugacy invariant length function on $(\bigoplus_B K) \wr_B \text{Sym}(B)$ defined in Proposition 2.2. So

$$\tilde{\ell}((k_b)_{b \in B}, \pi) = d_{\text{Hamm}}(\pi, 1) + \frac{1}{|B|} \sum_{\substack{b \in B \\ \pi(b)=b}} \ell'(k_b).$$

We let \tilde{d} be the bi-invariant metric on $(\bigoplus_B K) \wr_B \text{Sym}(B)$ induced by $\tilde{\ell}$. Our aim is to prove the following.

Proposition 3.1. *Let $F \subseteq G \wr H$ be finite and $\varepsilon > 0$. There are finite sets $E_G \subseteq G$ and $E, E_H \subseteq H$, and an $\varepsilon' > 0$ with the following properties. Let*

- $\sigma: H \rightarrow \text{Sym}(B)$ be an (E_H, ε') -sofic approximation,
- K be a group with a conjugacy invariant length function ℓ with $\ell \leq 1$,
- $\theta: G \rightarrow K$ be a map for some group K .

Then $\Theta: G \wr H \rightarrow (\bigoplus_B K) \wr_B \text{Sym}(B)$, as constructed above, has the following properties.

(a) *Suppose ℓ' is any conjugacy invariant length function on $\bigoplus_B K$ which restricts to ℓ on each copy of K .*

If $\theta: G \rightarrow K$ is (E_G, ε') -multiplicative, then Θ is (F, ε) -multiplicative.

(b) *Suppose $\ell' = \ell_{\max}$, which is defined on $\bigoplus_B K$ by*

$$\ell_{\max}((k_b)_{b \in B}) = \max_{b \in B} \ell(k_b).$$

Suppose $c: G \setminus \{1\} \rightarrow (0, \infty)$ is given. Define $c': (G \wr H) \setminus \{1\} \rightarrow (0, \infty)$ by

$$c'(g, h) = \begin{cases} \frac{1}{2}, & \text{if } h \neq 1 \\ \max_{x \in \text{Supp}(g)} \frac{1}{2} c(g_x), & \text{if } h = 1, g = (g_x)_{x \in H}. \end{cases}$$

Then, if $\theta: G \rightarrow K$ is (E_G, c) -injective then Θ is (F, c') -injective.

Remark 3.2. In each case, the almost multiplicativity or almost injectivity of Θ are given with respect to the metric \tilde{d} . In the second part, we will denote this metric \tilde{d}_{\max} , so as to emphasise the particular length function $\ell' = \ell_{\max}$ that has been used.

The remainder of this section is dedicating to proving Proposition 3.1. We will see below that the following upper bounds on ε' are sufficient:

$$(1) \quad \begin{aligned} \text{for (a): } \quad & \varepsilon' < \frac{\varepsilon}{48|E|^2}, \\ \text{for (b): } \quad & \varepsilon' < \frac{1}{16|E|^2} \min \{c(g), 1 \mid g \in E_G \setminus \{1\}\}. \end{aligned}$$

As we see, the bounds on ε' depend only on ε and the set F .

We remark that $\Theta(1, 1) = 1$ by construction. We first explain how to define the sets E , E_G and E_H .

Let $F \subseteq G \wr H$ be finite and $\varepsilon > 0$. Define projections $\text{proj}_G: G \wr H \rightarrow \bigoplus_H G$ and $\text{proj}_H: G \wr H \rightarrow H$ by $\text{proj}_G(g, h) = g$ and $\text{proj}_H(g, h) = h$. Let E_1, E_2 be as in Lemma 2.1 for the finite set $F_0 = F \cup \{1\} \cup F^{-1}$. As in the proof of Lemma 2.1, we have

$$\begin{aligned} E_1 &= \{\alpha_h(g) : h \in \text{proj}_H(F_0), g \in \text{proj}_G(F_0)\}, \\ E_2 &= \text{proj}_H(F_0). \end{aligned}$$

Recall that for $g = (g_x)_{x \in H} \in \bigoplus_H G$ the support of g , denoted $\text{Supp}(g)$, is the set of $x \in H$ with $g_x \neq 1$. We set

$$\begin{aligned} E &= E_2 \cup \bigcup_{\substack{g \in E_1 \\ h \in E_2}} h \text{Supp}(g), \\ E_G &= \{g_x \in G : (g_x)_{x \in H} \in E_1\}, \\ E_H &= E^{-1}E. \end{aligned}$$

Then our collections of finite sets satisfy the following properties, each of which we need later on:

$$\begin{aligned} E_G &\supseteq \{g_x : (g_x) \in E_1, x \in E\}, \\ E &\supseteq h \text{Supp}(g) \text{ for all } h \in E_2, g \in E_1, \\ E &\supseteq E_2, \\ E_H &\supseteq E \cup E^{-1} \cup E^{-1}E. \end{aligned}$$

Let K be a group with a conjugacy invariant length function ℓ and let $\theta: G \rightarrow K$ be (E_1, ε') -multiplicative and (E_1, c) -injective, where ε' is controlled by the bounds in (1) above. Let $\sigma: H \rightarrow \text{Sym}(B)$ be a (E_H, ε') -sofic approximation. Recall the set B_E is defined from E as the intersections of sets B_1, B_2 (which depend only on E). Lemma 3.3 confirms that, since σ is a sofic approximation, B_E makes up a significant proportion of the set B . Use the set $E \subseteq H$ and the maps θ, σ to define the maps θ_B, Θ , as constructed at the start of this section.

3.3. Part (a) of Proposition 3.1. We claim that if ε' is sufficiently small, then the map Θ is (F, ε) -multiplicative. We first make the following preliminary observation.

Lemma 3.3. *Let $\kappa > 0$. If $\varepsilon' < \frac{\kappa}{4|E|^2}$ then $|B_E| \geq (1 - \kappa)|B|$.*

Proof. Note that

$$B_1 = \bigcap_{\substack{h_1, h_2 \in E \\ h_1 \neq h_2}} \{b \in B : \sigma(h_1)^{-1}b \neq \sigma(h_2)^{-1}b\},$$

$$B_2 = \bigcap_{\substack{h_1, h_2 \in E \\ h_1 \neq h_2}} \{b \in B : \sigma(h_2)^{-1}\sigma(h_1)^{-1}b = \sigma(h_1h_2)^{-1}b\}.$$

So

$$\frac{|B \setminus B_1|}{|B|} \leq \sum_{\substack{h_1, h_2 \in E \\ h_1 \neq h_2}} \left(1 - \frac{|\{b \in B : \sigma(h_1)^{-1}b \neq \sigma(h_2)^{-1}b\}|}{|B|}\right).$$

For all $h_1, h_2 \in E$ with $h_1 \neq h_2$ we have

$$\begin{aligned} \frac{|\{b \in B : \sigma(h_1)^{-1}b \neq \sigma(h_2)^{-1}b\}|}{|B|} &= d_{\text{Hamm}}(\sigma(h_1), \sigma(h_2)) \\ &= d_{\text{Hamm}}(\sigma(h_2)^{-1}\sigma(h_1), 1). \end{aligned}$$

Since d_{Hamm} is a invariant under left multiplication, and $E_H \supseteq E \cup E^{-1}$ we have that

$$d_{\text{Hamm}}(\sigma(h_2)^{-1}, \sigma(h_2^{-1})) = d_{\text{Hamm}}(1, \sigma(h_2)\sigma(h_2^{-1})) < \varepsilon'$$

by the (E_H, ε') -soficity of σ . Inserting this into the above two inequalities and using that d_{Hamm} is invariant under right multiplication we see that

$$\begin{aligned} \frac{|\{b \in B : \sigma(h_1)^{-1}b \neq \sigma(h_2)^{-1}b\}|}{|B|} &= d_{\text{Hamm}}(\sigma(h_1), \sigma(h_2)) \\ &= d_{\text{Hamm}}(\sigma(h_2)^{-1}\sigma(h_1), 1) \\ &> d_{\text{Hamm}}(\sigma(h_2^{-1})\sigma(h_1), 1) - \varepsilon' \\ &\geq d_{\text{Hamm}}(\sigma(h_2^{-1}h_1), 1) - 2\varepsilon' \\ &> 1 - 3\varepsilon', \end{aligned}$$

where in the last two lines we again use that $E_H \supseteq E \cup E^{-1} \cup E^{-1}E$. Thus

$$\frac{|B_1|}{|B|} \geq (1 - 3|E|^2\varepsilon').$$

Similarly, (E_H, ε') -multiplicativity of σ gives

$$\frac{|B_2|}{|B|} \geq 1 - \sum_{h_1, h_2 \in E} \left(1 - d_{\text{Hamm}}(\sigma(h_1h_2), \sigma(h_1)\sigma(h_2))\right) \geq 1 - |E|^2\varepsilon'.$$

This proves the Lemma. \square

For (a), take $\kappa > 0$ so that $\kappa < \frac{\varepsilon}{12}$, and take $\varepsilon' > 0$ satisfying the hypothesis of Lemma 3.3, so we will have $\varepsilon' < \frac{\varepsilon}{48|E|^2}$.

We now show that Θ is (F, ε) -multiplicative with respect to the metric \tilde{d} as defined from ℓ' using Proposition 2.2, thus completing the proof of (a) of Proposition 3.1.

We apply Lemma 2.1, verifying below the four necessary conditions. We first check that the restriction to $\bigoplus_H G$ is $(E_1, \varepsilon/6)$ -multiplicative. Recall that throughout Proposition 3.1 we assume that $\ell \leq 1$, while in part (a) we assume furthermore that ℓ' restricts to ℓ on each copy of K . Let $g, g' \in E_1$ with $g = (g_x)_{x \in H}, g' = (g'_x)_{x \in H}$. Then

$$\begin{aligned} \tilde{d}(\theta_B(g)\theta_B(g'), \theta_B(gg')) &= \frac{1}{|B|} \sum_{b \in B} \ell'(\theta_b(gg')^{-1}\theta_b(g)\theta_b(g')) \\ &\leq \kappa + \frac{1}{|B|} \sum_{b \in B_E} \ell'(\theta_b(gg')^{-1}\theta_b(g)\theta_b(g')). \end{aligned}$$

By the definitions of θ_b and of E_1 , we realise that each component of $\theta_b(gg')^{-1}\theta_b(g)\theta_b(g')$ is either 1 or $\theta(g_x g'_x)^{-1}\theta(g_x)\theta(g'_x)$, for $x \in E$. Thus

$$\begin{aligned} \tilde{d}(\theta_B(g)\theta_B(g'), \theta_B(gg')) &\leq \kappa + \frac{1}{|B|} \sum_{b \in B_E} \sum_{x \in E} \ell(\theta(g_x g'_x)^{-1}\theta(g_x)\theta(g'_x)) \\ &\leq \kappa + |E| \varepsilon', \end{aligned}$$

where in the last line we use that θ is (E_G, ε') -multiplicative. Since $\kappa + |E| \varepsilon' < \frac{\varepsilon}{6}$, we see that Θ is $(E_1, \varepsilon/6)$ -multiplicative.

The fact that the restriction to H is $(E_2, \varepsilon/6)$ -multiplicative is more straight-forward. Indeed, for $h, h' \in E_2$ we have

$$\tilde{d}(\sigma(hh'), \sigma(h)\sigma(h')) = d_{\text{Hamm}}(\sigma(hh'), \sigma(h)\sigma(h')) < \varepsilon',$$

where we note that we can use the multiplicative property of σ since $E_2 \subseteq E_H$.

By construction, the third condition of Lemma 2.1, bounding the distance between $\Theta(g, h)$ and $\Theta(g, 1)\Theta(1, h)$, is automatically satisfied by Θ , since these elements are equal.

We finish part (a) by verifying the bound on $\tilde{d}(\Theta(1, h)\Theta(g, 1), \Theta(\alpha_h(g), 1)\Theta(1, h))$ for $g \in E_1, h \in E_2$. We have

$$\begin{aligned} \tilde{d}(\Theta(1, h)\Theta(g, 1), \Theta(\alpha_h(g), 1)\Theta(1, h)) &= \tilde{d}((\alpha_{\sigma(h)}(\theta_B(g)), \sigma(h)), (\theta_B(\alpha_h(g)), \sigma(h))) \\ &= \frac{1}{|B|} \sum_{b \in B} \ell'(\theta_{\sigma(h)^{-1}b}(g)^{-1}\theta_b(\alpha_h(g))) \\ &= \frac{1}{|B|} \sum_{b \in B} \ell'(\theta_b(g)^{-1}\theta_{\sigma(h)b}(\alpha_h(g))) \end{aligned}$$

Using Lemma 3.3, and that $\ell' \leq 1$, we can disregard what happens for b outside of both B_E and $\sigma(h)^{-1}B_E$ for a controlled cost. This gives us the following upper bound for the above distance:

$$2\kappa + \frac{1}{|B|} \sum_{b \in B_E \cap \sigma(h)^{-1}B_E} \ell'(\theta_b(g)^{-1}\theta_{\sigma(h)b}(\alpha_h(g))).$$

Since $\text{Supp}(\alpha_h(g)) = h \text{Supp}(g)$, and E contains both $\text{Supp}(g)$ and $h \text{Supp}(g)$, it follows that for every $b \in B_E \cap \sigma(h)^{-1}(B_E)$ we have

$$\theta_{\sigma(h)b}(\alpha_h(g)) = \prod_{x \in h \text{Supp}(g)} \theta_{\sigma(h)b}^{(x)}(g_{h^{-1}x}) = \prod_{x \in \text{Supp}(g)} \theta_{\sigma(h)b}^{(hx)}(g_x).$$

Note that we have used that $\theta(1) = 1$ to restrict the number of terms in the product. We use that for $h \in E$ (and hence for $h \in E_2$) and $b \in B_E \cap \sigma(h)^{-1}B_E$ we have that $\theta_{\sigma(h)b}^{(hx)}(g) = \theta_b^{(x)}(g)$.

Inserting this into the above equation we see that

$$\theta_{\sigma(h)b}(\alpha_h(g)) = \prod_{x \in \text{Supp}(g)} \theta_b^{(x)}(g_x) = \theta_b(g).$$

Returning to the above inequality, we have shown that

$$\frac{1}{|B|} \sum_{b \in B_E \cap \sigma(h)^{-1} B_E} \ell'(\theta_b(g)^{-1} \theta_{\sigma(h)b}(\alpha_h(g))) = 0$$

so

$$\tilde{d}(\Theta(1, h)\Theta(g, 1), \Theta(\alpha_h(g), 1)\Theta(1, h)) < 2\kappa < \frac{\varepsilon}{6}.$$

The hypotheses of Lemma 2.1 have been checked, completing the proof of part (a) of Proposition 3.1.

3.4. Part (b) of Proposition 3.1. We now show that Θ is (F, c') -injective, when the length function ℓ' on $\bigoplus_B K$ is ℓ_{\max} . Let \tilde{d}_{\max} be the metric on $(\bigoplus_B K) \wr_B \text{Sym}(B)$ constructed from $\ell' = \ell_{\max}$ using Proposition 2.2.

In order to get (b) we will need to further restrict the size of κ (and hence also of ε'). We take κ small enough so that we also have

$$\kappa < \frac{1}{4} \min \{c(g), 1 \mid g \in E_G \setminus \{1\}\}.$$

First suppose $(g, h) \in F$. If $h \neq 1$, then

$$\tilde{d}_{\max}((\theta_B(g), \sigma(h)), (1, 1)) \geq d_{\text{Hamm}}(\sigma(h), 1) \geq 1 - \varepsilon' \geq 1/2 = c'(g, h).$$

We may therefore assume that $h = 1$. Let $g = (g_x)_{x \in E}$. We then have that

$$\begin{aligned} \tilde{d}_{\max}((\theta_B(g), 1), (1, 1)) &= \frac{1}{|B|} \sum_{b \in B} \ell_{\max}(\theta_b(g)) \\ &\geq -\kappa + \frac{1}{|B|} \sum_{b \in B_E} \ell_{\max}(\theta_b(g)). \end{aligned}$$

Since the components of $\theta_b(g)$ are either 1 or $\theta(g_x)$ for some $x \in E$, and this will be the case for each $b \in B_E$, we get $\ell_{\max}(\theta_b(g)) = \max_{x \in E} \ell(\theta(g_x))$. Hence

$$\begin{aligned} \tilde{d}_{\max}((\theta_B(g), 1), (1, 1)) &\geq -\kappa + \frac{|B_E|}{|B|} \max_{x \in E} \ell(\theta(g_x)) \\ &\geq -\kappa + (1 - \kappa) \max_{x \in \text{Supp}(g)} c(g_x) \end{aligned}$$

where the last inequality follows from Lemma 3.3 and the fact that θ is (E_G, c) -injective. By the choices of κ and E_G , we get

$$-\kappa + (1 - \kappa) \max_{x \in \text{Supp}(g)} c(g_x) \geq \frac{-1}{4} \max_{x \in \text{Supp}(g)} c(g_x) + \left(1 - \frac{1}{4}\right) \max_{x \in \text{Supp}(g)} c(g_x) = c'(g, 1).$$

This verifies that Θ is (F, c') -injective, and thus completes the proof of Proposition 3.1.

Remark 3.4. Our proof can in fact be subtly modified to give a stronger version of Proposition 3.1, that is reminiscent of the notion of strong discrete \mathcal{C} -approximations of Holt-Rees [14]. Namely, for any $\eta > 0$ we can improve the conclusion of part (b) to say that Θ is (F, c') -injective, where c' is given by

$$c'(g, h) = \begin{cases} (1 - \eta), & \text{if } h \neq 1 \\ \max_{x \in \text{Supp}(g)} (1 - \eta)c(g_x), & \text{if } h = 1, g = (g_x)_{x \in H}. \end{cases}$$

For this improved version, the parameters $E, E_H, E_G, \varepsilon'$ will depend upon η . We have elected to not give this improved version in order to simplify the statement of the proposition and its proof.

4. APPLICATIONS OF PROPOSITION 3.1

In this section, we use Proposition 3.1 to prove Theorem 3. Part (iv) of Theorem 3 follows immediately from Proposition 3.1, so we focus on proving the remaining three parts. Each of parts (i),(ii),(iii) are proved below in separate subsections. We recall that the aim is to show that, for H a countable, discrete, sofic group, the wreath product $G \wr H$ is respectively sofic, hyperlinear, or linear sofic, whenever G is such a group.

4.1. Proof of Part (i): Sofic. We restate and prove our soficity result for wreath products.

Theorem 4.1. *Let G, H be countable, discrete, sofic groups. Then $G \wr H$ is sofic.*

Proof. In order to show that $G \wr H$ is sofic, we show that $G \wr H$ is \mathcal{C} -approximable, where \mathcal{C} is the class of symmetric groups with the normalized Hamming distance. To do this we compose the map Θ from Proposition 3.1 with a second map Ψ , as described below.

Let $F \subseteq G \wr H$ be finite and $\varepsilon > 0$. Let E_G, E, E_H and $\varepsilon' > 0$ be as in Proposition 3.1 for F, ε . Define c on $G \wr H \setminus \{1\}$ by $c(g) = \frac{1}{2}$, and let

$$c': G \wr H \setminus \{1\} \rightarrow (0, 1/2]$$

be the map constructed in Proposition 3.1. Thus $c'(g, h)$ is either $1/2$ if $h \neq 1$, or $1/4$ otherwise.

Since G, H are sofic we can find corresponding sofic approximations. For H we take $\sigma: H \rightarrow \text{Sym}(B)$, for a finite set B , to be an (E_H, ε') -sofic approximation; for G we take $\theta: G \rightarrow \text{Sym}(A)$, for a finite set A , to be an (E_G, ε') -sofic approximation. Note that, since $\varepsilon' < 1/2$ (see (1) following Proposition 3.1), the (E_G, ε') -free condition of θ implies that it is (E_G, c) -injective.

With these maps, let $\Theta: G \wr H \rightarrow (\bigoplus_B \text{Sym}(A)) \wr_B \text{Sym}(B)$ be the map constructed in Section 3, with $K = \text{Sym}(A)$. We now explain how we embed $(\bigoplus_B \text{Sym}(A)) \wr_B \text{Sym}(B)$ into $\text{Sym}(\bigoplus_B A \oplus B)$. First, define

$$\Phi: \bigoplus_B \text{Sym}(A) \rightarrow \text{Sym}\left(\bigoplus_B A\right)$$

by the diagonal action

$$\Phi((\pi_\beta)_{\beta \in B}): (a_\beta)_{\beta \in B} \mapsto (\pi_\beta(a_\beta))_{\beta \in B}, \quad \text{for } \pi_\beta \in \text{Sym}(A), (a_\beta)_{\beta \in B} \in \bigoplus_B A.$$

Then, use Φ to define the embedding

$$\Psi: \left(\bigoplus_B \text{Sym}(A)\right) \wr_B \text{Sym}(B) \rightarrow \text{Sym}\left(\bigoplus_B A \oplus B\right)$$

by

$$\Psi((\pi_b)_{b \in B}, \tau): (a, \beta) \mapsto (\Phi(\pi_\beta)(a), \tau(\beta))$$

for $\pi_b \in \bigoplus_B \text{Sym}(A)$, $\tau \in \text{Sym}(B)$, $a \in \bigoplus_B A$, and $\beta \in B$.

Let ℓ', ℓ_{\max} be the conjugacy invariant length functions on $\bigoplus_B \text{Sym}(A)$ given by

$$\begin{aligned} \ell'(\pi) &= d_{\text{Hamm}}(\Phi(\pi), 1), \\ \ell_{\max}(\pi) &= \max_{\beta \in B} d_{\text{Hamm}}(\pi_\beta, 1) \end{aligned}$$

for $\pi = (\pi_b)_{b \in B} \in \bigoplus_B \text{Sym}(A)$. Then take $\tilde{d}, \tilde{d}_{\max}$ to be the bi-invariant metrics as constructed in Proposition 2.2 from the length functions ℓ', ℓ_{\max} on $\bigoplus_B \text{Sym}(A)$.

A simple computation shows that for $\pi_1, \pi_2 \in \bigoplus_B (\bigoplus_B \text{Sym}(A))$, $\tau_1, \tau_2 \in \text{Sym}(B)$, we have:

$$(2) \quad d_{\text{Hamm}}(\Psi(\pi_1, \tau_2), \Psi(\pi_2, \tau_2)) = \tilde{d}((\pi_1, \tau_1), (\pi_2, \tau_2)).$$

It thus follows directly from Proposition 3.1 that $\Psi \circ \Theta$ is (F, ε) -multiplicative with respect to the Hamming distance.

We now show that $\Psi \circ \Theta$ is (F, c') -injective. Let $\pi \in \bigoplus_B (\bigoplus_B \text{Sym}(A))$ and $\tau \in \text{Sym}(B)$. Write $\pi = (\pi_b)_{b \in B}$ for $\pi_b \in \bigoplus_B \text{Sym}(A)$ and, for a fixed $b \in B$, let $\pi_b = (\pi_{b,\beta})_{\beta \in B}$. For each $b \in B$ such that $\tau(b) = b$ we then have

$$\begin{aligned} d_{\text{Hamm}}(\Phi(\pi_b), 1) &= 1 - \frac{1}{|A|^{|B|}} |\{(a_\beta)_{\beta \in B} \mid \pi_{b,\beta} a_\beta = a_\beta\}| \\ &= 1 - \frac{1}{|A|^{|B|}} \prod_{\beta \in B} |\{a \in A \mid \pi_{b,\beta} a = a\}| \\ &= 1 - \prod_{\beta \in B} (1 - d_{\text{Hamm}}(\pi_{b,\beta}, 1)) \end{aligned}$$

which implies

$$\tilde{d}((\pi, \tau), (1, 1)) = d_{\text{Hamm}}(\tau, 1) + \frac{1}{|B|} \sum_{\substack{b \in B \\ \tau(b)=b}} \left[1 - \prod_{\beta \in B} (1 - d_{\text{Hamm}}(\pi_{b,\beta}, 1)) \right].$$

Since $0 \leq d_{\text{Hamm}}(\pi_{b,\beta}, 1) \leq 1$ we have for each $b \in B$:

$$\prod_{\beta \in B} (1 - d_{\text{Hamm}}(\pi_{b,\beta}, 1)) \leq 1 - \max_{\beta \in B} d_{\text{Hamm}}(\pi_{b,\beta}, 1)$$

and inserting this into the above expression for \tilde{d} shows that

$$\tilde{d}((\pi, \tau), (1, 1)) \geq \tilde{d}_{\max}((\pi, \tau), (1, 1)).$$

Combining equation (2) with the preceding inequality, we get for each $g \in F \setminus \{1\}$

$$d_{\text{Hamm}}(\Psi(\Theta(g)), 1) = \tilde{d}(\Theta(g), (1, 1)) \geq \tilde{d}_{\max}(\Theta(g), (1, 1)) \geq c'(g)$$

since Proposition 3.1 implies Θ is (F, c') -injective with respect to \tilde{d}_{\max} . This shows that $\Psi \circ \Theta$ is (F, c') -injective with respect to the Hamming distance. Hence we have shown that $G \wr H$ is \mathcal{C} -approximable, where \mathcal{C} is the class of symmetric groups equipped with the Hamming distance and this means that $G \wr H$ is sofic. \square

We remark that one can use the improved version of Proposition 3.1, as per Remark 3.4, to show that $\Psi \circ \Theta$ as considered in the above proof is an (F, ε) -sofic approximation provided $\theta: G \rightarrow \text{Sym}(A)$ and $\sigma: H \rightarrow \text{Sym}(B)$ are sufficiently good sofic approximations. In this way one can in fact directly show that $G \wr H$ has arbitrarily good sofic approximations.

4.2. Proof of Part (ii): Hyperlinear. In this section, we deduce hyperlinearity of $G \wr H$, assuming that G is hyperlinear and H is sofic. Hyperlinear groups are defined by admitting a metric approximation to unitary groups, $\mathcal{U}(n)$, paired with the normalized Hilbert-Schmidt metric.

Let $\text{tr}: M_n(\mathbb{C}) \rightarrow \mathbb{C}$ be the normalized trace:

$$\text{tr}(A) = \frac{1}{n} \sum_{j=1}^n A_{jj}$$

where $A = (A_{ij}) \in M_n(\mathbb{C})$.

Definition 7. The *normalized Hilbert-Schmidt norm* on $M_n(\mathbb{C})$ is defined by

$$\|A\|_2 = \text{tr}(A^*A)^{1/2}, \quad \text{for } A \in M_n(\mathbb{C}).$$

The *normalized Hilbert-Schmidt metric* on $\mathcal{U}(n)$ is therefore given by

$$d_{\text{HS}}(U, V) = \|U - V\|_2, \quad \text{for } U, V \in \mathcal{U}(n).$$

Definition 8. We say a group is *hyperlinear* if it is \mathcal{C} -approximable, where \mathcal{C} is the class of unitary groups, paired with the normalized Hilbert-Schmidt metrics.

We will need that our approximations $\theta: G \rightarrow \mathcal{U}(n)$ not only map $\theta(g)$ far away from Id for $g \neq 1$, but that in fact $\theta(g)$ is far away from the unit circle $S^1 = \{\lambda \text{Id} : |\lambda| = 1\}$ in $\mathcal{U}(n)$. To put this in a framework where we can take advantage of Proposition 3.1, we use the following set-up.

Define \bar{d}_{HS} , a bi-invariant metric on $\mathcal{U}(n)/S^1$, by

$$\bar{d}_{\text{HS}}(US^1, VS^1) = \inf_{\lambda \in S^1} d_{\text{HS}}(\lambda U, V), \quad \text{for } U, V \in \mathcal{U}(n).$$

We will abuse this notation and write $\bar{d}_{\text{HS}}(U, V)$. Note that we can directly use the normalized trace to calculate $\bar{d}_{\text{HS}}(U, \text{Id})$ as follows:

$$\bar{d}_{\text{HS}}(U, \text{Id})^2 = \inf_{\lambda \in S^1} \|U - \lambda \text{Id}\|_2^2 = \inf_{\lambda \in S^1} 2 - 2 \text{re}(\bar{\lambda} \text{tr}(U)) = 2 - 2 |\text{tr}(U)|.$$

In light of this, we get the following reformulation of a result of Rădulescu in [22] which gives an equivalent definition of hyperlinearity.

Proposition 4.2. *Let G be a group and $c: G \setminus \{1\} \rightarrow (0, \sqrt{2})$ any function.*

Then G is hyperlinear if and only if for every $\varepsilon > 0$ and any finite $F \subseteq G$ there is a positive integer n and a function $\theta: G \rightarrow \mathcal{U}(n)$ which is (F, ε) -multiplicative and so that $q \circ \theta$ is (F, c) -injective, where $q: \mathcal{U}(n) \rightarrow \mathcal{U}(n)/S^1$ is the quotient map.

Theorem 4.3. *Let H be a countable, discrete, sofic group and G a countable, discrete, hyperlinear group. Then $G \wr H$ is hyperlinear.*

Proof. We proceed in an analogous manner as for Theorem 4.1, when we dealt with soficity. In particular, we show that $G \wr H$ is \mathcal{C} -approximable, where \mathcal{C} is as in Definition 8. The necessary maps to demonstrate this will be constructed as a composition, starting with Θ from Proposition 3.1 followed by an appropriate embedding into a unitary group.

Step 1: Setting the scene.

Let $F \subseteq G \wr H$ be finite and $\varepsilon > 0$. Let E_G, E, E_H and $\varepsilon' > 0$ be as Proposition 3.1 for F, ε . Let $c: G \setminus \{1\} \rightarrow (0, 1/2]$ be given by $c(g) = \frac{1}{2}$ for $g \in G \setminus \{1\}$ and let $c': G \wr H \setminus \{1\} \rightarrow (0, 1/2]$ be the map constructed in Proposition 3.1. Since H is sofic we can find an (E_H, ε') -sofic approximation $\sigma: H \rightarrow \text{Sym}(B)$ for some finite set B . Since G is hyperlinear we apply Proposition 4.2 to find an (E_G, ε) -multiplicative map $\theta: G \rightarrow \mathcal{U}(\mathcal{H})$ for some finite-dimensional Hilbert space \mathcal{H} so that $q \circ \theta$ is (E_G, c) -injective.

Let $\Theta: G \wr H \rightarrow (\bigoplus_B \mathcal{U}(\mathcal{H})) \wr_B \text{Sym}(B)$ be the map constructed from θ, σ and E in Section 3. Similarly construct $\bar{\Theta}: G \wr H \rightarrow (\bigoplus_B \mathcal{U}(\mathcal{H})/S^1) \wr_B \text{Sym}(B)$ from $q \circ \theta, \sigma$ and E .

Define

$$\Phi: \bigoplus_B \mathcal{U}(\mathcal{H}) \rightarrow \mathcal{U}(\mathcal{H}^{\otimes B})$$

by

$$\Phi: (V_\beta)_{\beta \in B} \mapsto \bigotimes_{\beta \in B} V_\beta, \text{ for } (V_\beta)_{\beta \in B} \in \bigoplus_B \mathcal{U}(\mathcal{H}).$$

We now define

$$\Psi: \left(\bigoplus_B \mathcal{U}(\mathcal{H}) \right) \wr_B \text{Sym}(B) \rightarrow \mathcal{U} \left(\bigoplus_B (\mathcal{H}^{\otimes B}) \right)$$

by

$$\Psi((U_b)_{b \in B}, \tau): (\xi_b)_{b \in B} \mapsto (\Phi(U_b)(\xi_{\tau^{-1}(b)}))_{b \in B}$$

for $(\xi_b)_{b \in B} \in \bigoplus_B (\mathcal{H}^{\otimes B})$, $(U_b)_{b \in B} \in \bigoplus_B \bigoplus_B \mathcal{U}(\mathcal{H})$, and $\tau \in \text{Sym}(B)$. The collection of maps we have is summarized in Figure 1.

$$\begin{array}{ccccc} G \wr H & \xrightarrow{\Theta} & \left(\bigoplus_B \mathcal{U}(\mathcal{H}) \right) \wr_B \text{Sym}(B) & \xrightarrow{\Psi} & \mathcal{U} \left(\bigoplus_B (\mathcal{H}^{\otimes B}) \right) \\ & \searrow \Theta & \downarrow & & \\ & & \left(\bigoplus_B \mathcal{U}(\mathcal{H})/S^1 \right) \wr_B \text{Sym}(B) & & \end{array}$$

FIGURE 1. A plan of the maps involved.

Let $\tilde{d}, \tilde{d}_{\max}$ be the bi-invariant metrics on $(\bigoplus_B \mathcal{U}(\mathcal{H})) \wr_B \text{Sym}(B)$ and $(\bigoplus_B \mathcal{U}(\mathcal{H})/S^1) \wr_B \text{Sym}(B)$, respectively, induced by Proposition 2.2 from the length functions ℓ', ℓ_{\max} on $\bigoplus_B \mathcal{U}(\mathcal{H})$ and $\bigoplus_B \mathcal{U}(\mathcal{H})/S^1$, respectively, which are given by

$$\ell'(V) = \frac{1}{\sqrt{2}} \|\Phi(V) - \text{Id}\|_2,$$

$$\ell_{\max}(\bar{V}) = \max_{\beta \in B} \frac{\bar{d}_{\text{HS}}(q(V_\beta), 1)}{\sqrt{2}},$$

for $V = (V_\beta)_{\beta \in B} \in \bigoplus_B \mathcal{U}(\mathcal{H})$, and $\bar{V} = (q(V_\beta))_{\beta \in B} \in \bigoplus_B \mathcal{U}(\mathcal{H})/S^1$.

Step 2: A formula for $d_{\text{HS}}(\Psi(U, \tau), \text{Id})$.

We aim to bound the d_{HS} -distance from a point in the image of Ψ to the identity in terms of the \tilde{d} -distance for its pre-image. To this end, we first observe that the matrix representation for $\Psi(U, \tau)$ will be a block permutation matrix, with blocks corresponding to elements of B . The matrix will have a non-zero block in the (b, b) -position precisely when $\tau(b) = b$. Thus we get

$$\text{tr}(\Psi(U, \tau)) = \frac{1}{|B|} \sum_{\substack{b \in B \\ \tau(b)=b}} \text{tr}(\Phi(U_b)), \text{ where } U = (U_b)_{b \in B}.$$

This implies that

$$\|\Psi(U, \tau) - \text{Id}\|_2^2 = 2 - 2 \text{re}(\text{tr}(\Psi(U, \tau))) = 2 - \frac{2}{|B|} \sum_{\substack{b \in B \\ \tau(b)=b}} \text{re}(\text{tr}(U_b)).$$

By the definition of the Hamming metric we can rewrite the right-hand side as

$$2d_{\text{Hamm}}(\tau, 1) + \frac{2}{|B|} \sum_{\substack{b \in B \\ \tau(b)=b}} 1 - \text{re}(\text{tr}(\Phi(U_b))).$$

Hence

$$(3) \quad \|\Psi(U, \tau) - \text{Id}\|_2^2 = 2d_{\text{Hamm}}(\tau, 1) + \frac{2}{|B|} \sum_{\substack{b \in B \\ \tau(b)=b}} \|\Phi(U_b) - \text{Id}\|_2^2.$$

Step 3: Almost multiplicativity.

Since $\|\Phi(U_b) - \text{Id}\|_2 \leq \sqrt{2}$, we can get an upper bound of

$$\|\Psi(U, \tau) - \text{Id}\|_2^2 \leq 2d_{\text{Hamm}}(\tau, 1) + \frac{2\sqrt{2}}{|B|} \sum_{\substack{b \in B \\ \tau(b)=b}} \|\Phi(U_b) - \text{Id}\|_2 \leq 4\tilde{d}((U, \tau), 1).$$

In summary, since Ψ is a homomorphism, for $(U_1, \tau_1), (U_2, \tau_2) \in (\bigoplus_B \mathcal{U}(\mathcal{H})) \wr_B \text{Sym}(B)$ we have shown

$$d_{\text{HS}}(\Psi(U_1, \tau_1), \Psi(U_2, \tau_2)) \leq 2\tilde{d}((U_1, \tau_1), (U_2, \tau_2))^{1/2}.$$

From Proposition 3.1 we know that Θ is (F, ε) -multiplicative. With this, the above inequality then implies that $\Psi \circ \Theta$ is $(F, 2\sqrt{\varepsilon})$ -multiplicative with respect to d_{HS} .

Step 4: Almost injectivity.

Let $V = (V_\beta)_{\beta \in B} \in \bigoplus_B \mathcal{U}(\mathcal{H})$. For each β , we have that

$$|\text{tr}(V_\beta)| = 1 - \left(\frac{\bar{d}_{\text{HS}}(q(V_\beta), \text{Id})}{\sqrt{2}} \right)^2.$$

Thus

$$|\text{tr}(\Phi(V))| = \prod_{\beta \in B} |\text{tr}(V_\beta)| \leq 1 - \max_{\beta \in B} \left(\frac{\bar{d}_{\text{HS}}(q(V_\beta), \text{Id})}{\sqrt{2}} \right)^2 = 1 - \ell_{\max}(\bar{V})^2$$

where $\bar{V} = (q(V_\beta))_{\beta \in B}$. Since $2 - 2|\text{tr}(\Phi(V))| \leq \|\Phi(V) - \text{Id}\|_2^2$, we get $\ell_{\max}(\bar{V})^2 \leq \frac{1}{2} \|\Phi(V) - \text{Id}\|_2^2$. Inserting this into equation (3) and arguing as in Section 4.1 we see that, if $\bar{U}_b = (q(U_{b,\beta}))_{\beta \in B}$, then

$$\begin{aligned} \|\Psi(U, \tau) - \text{Id}\|_2^2 &\geq 2d_{\text{Hamm}}(\tau, 1) + \frac{2}{|B|} \sum_{\substack{b \in B \\ \tau(b)=b}} \ell_{\max}(\bar{U}_b)^2 \\ &\geq 2d_{\text{Hamm}}(\tau, 1)^2 + 2 \left(\frac{1}{|B|} \sum_{\substack{b \in B \\ \tau(b)=b}} \ell_{\max}(\bar{U}_b) \right)^2 \\ &\geq \left(d_{\text{Hamm}}(\tau, 1) + \frac{1}{|B|} \sum_{\substack{b \in B \\ \tau(b)=b}} \ell_{\max}(\bar{U}_b) \right)^2 \\ &= \tilde{d}_{\max}((\bar{U}, \tau), 1)^2 \end{aligned}$$

where $\bar{U} = (\bar{U}_b)_{b \in B}$. As $q \circ \theta$ is (E_G, c) -injective, it follows by Proposition 3.1 that $\bar{\Theta}$ is (F, c') -injective. Thus, for (U, τ) in the image of Θ , it follows that (\bar{U}, τ) is in the image of $\bar{\Theta}$, and

$$d_{\text{HS}}(\psi(U, \tau), \text{Id})^2 = \|\Psi(U, \tau) - \text{Id}\|_2^2 \geq (c'(x))^2.$$

Thus $\Psi \circ \Theta$ is (F, c') -injective and $(F, 2\sqrt{\varepsilon})$ -multiplicative. As $\varepsilon > 0$ is arbitrary the proof is complete. \square

As with soficity, one can use the improved version of Proposition 3.1 from Remark 3.4 to strengthen the bounds in the above results. In particular this will show that

$$\min_{x \in F \setminus \{1\}} \bar{d}_{\text{HS}}(\Psi(\Theta(x)), 1) \geq 1 - \varepsilon,$$

provided $\sigma: H \rightarrow \text{Sym}(B)$ is a sufficiently good sofic approximation and θ satisfies

$$\min_{g \in E} \bar{d}_{\text{HS}}(\theta(g), 1) > 1 - \kappa,$$

for a sufficiently large E and a sufficiently small κ . In this manner, we can directly verify the conclusion of Proposition 4.2 for $G \wr H$ if H is sofic and G is hyperlinear.

4.3. Proof of Part (iii): Linear Sofic. We recall the following definition due to Arzhantseva and Paunescu [1].

Definition 9. Let \mathbb{F} be a field. Define a bi-invariant metric d_{rk} on $\text{GL}_n(\mathbb{F})$ by

$$d_{\text{rk}}(A, B) = \frac{1}{n} \text{Rank}(A - B).$$

We say that a group is *linear sofic over \mathbb{F}* if it is \mathcal{C} -approximable, where \mathcal{C} consists of all general linear groups $\text{GL}_n(\mathbb{F})$, each paired with the metric d_{rk} .

In this section we use Proposition 3.1 to show that $G \wr H$ is linear sofic if G is linear sofic and H is sofic. Proving that the map we constructed is sufficiently injective turns out to be trickier than in any of the other cases. As in the case of hyperlinear groups, we will need that our linear sofic approximation $\theta: G \rightarrow \text{GL}_n(\mathbb{F})$ does not just satisfy that $\frac{1}{n} \text{Rank}(\theta(g) - \text{Id})$ is bounded away from 0 for $g \neq 1$, but in fact we need

$$\min_{\lambda \in \mathbb{K}^\times} \frac{1}{n} \text{Rank}_{\mathbb{K}}(\theta(g) - \lambda \text{Id}) > 0,$$

where $\text{Rank}_{\mathbb{K}}$ indicates that we are computing dimension over the algebraic closure \mathbb{K} of \mathbb{F} . Thus we use the following definition.

Definition 10. Let \mathbb{F} be a field, for $A, B \in \text{GL}_n(\mathbb{F})$, we let

$$\bar{d}_{\text{rk}}(A, B) = \min_{\lambda \in \mathbb{K}^\times} \frac{1}{n} \text{Rank}(A - \lambda B).$$

Note that, since $\frac{1}{n} \text{Rank}_{\mathbb{F}}(A - B) = \frac{1}{n} \text{Rank}_{\mathbb{K}}(A - B)$ for $A, B \in \text{GL}_n(\mathbb{F})$, we have $d_{\text{rk}}(A, B) \geq \bar{d}_{\text{rk}}(A, B)$.

We will then use the following fact, which is a consequence of an equivalent characterization of linear soficity given by Arzhantseva–Paunescu [1, Theorem 5.10].

Proposition 4.4. *Let G be a linear sofic group over the field \mathbb{F} and let \mathbb{K} denote the algebraic closure of \mathbb{F} .*

Then, for any $\delta \in (0, \frac{1}{8})$ and any finite $F \subseteq G$, there is a positive integer n and a function $\theta: G \rightarrow \text{GL}_n(\mathbb{F})$ which is (F, δ) -multiplicative with respect to d_{rk} , and so that $q \circ \theta$ is (F, c) -injective with respect to \bar{d}_{rk} , where $c(g) = \frac{1}{8} - \delta$ for all $g \in G$, and $q: \text{GL}_n(\mathbb{F}) \rightarrow \text{GL}_n(\mathbb{K})/\mathbb{K}^\times$

is the canonical map given by composing the natural inclusion $GL_n(\mathbb{F}) \rightarrow GL_n(\mathbb{K})$ with the quotient map $GL_n(\mathbb{K}) \rightarrow GL_n(\mathbb{K})/\mathbb{K}^\times$.

Proof. By [1, Theorem 5.10], it follows that there exists a function

$$\theta_0: G \rightarrow GL_m(\mathbb{F})$$

for some $m \in \mathbb{N}$, which is (F, δ) -multiplicative and so that $d_{\text{rk}}(\theta_0(g) - \text{Id}) \geq \frac{1}{4} - 2\delta$ for all $g \in F \setminus \{1\}$. Now consider

$$\theta: G \rightarrow GL_{2m}(\mathbb{F})$$

given in matrix block form by

$$\theta(g) = \begin{bmatrix} \theta_0(g) & 0 \\ 0 & \text{Id} \end{bmatrix}.$$

Fix $\lambda \in \mathbb{K}^\times$ and $g \in F \setminus \{1\}$. If $\lambda \neq 1$, then we see that $d_{\text{rk}}(\theta(g), \lambda \text{Id}) \geq \frac{1}{2}$. On the other hand, if $\lambda = 1$ then

$$\frac{1}{2m} \text{Rank}_{\mathbb{K}}(\theta(g) - \lambda \text{Id}) = \frac{1}{2} \cdot \left[\frac{1}{m} \text{Rank}_{\mathbb{F}}(\theta_0(g) - \text{Id}) \right] \geq \frac{1}{8} - \delta.$$

Thus θ is the required function. \square

In order to use Proposition 3.1 to prove that $G \wr H$ is linear sofic, we will need to use tensor products of matrices. The main fact we will need is that if $A \in GL_n(\mathbb{F})$, $B \in GL_k(\mathbb{F})$ and $\bar{d}_{\text{rk}}(A, \text{Id}), \bar{d}_{\text{rk}}(B, \text{Id})$ are both bounded away from zero, then $\bar{d}_{\text{rk}}(A \otimes B, \text{Id})$ is also bounded away from zero. We formulate this precisely in the Proposition 4.7 below, whose proof uses similar ideas to [1, Lemma 5.4, Prop 5.8].

Let $J_\alpha(A)$ denote the number of Jordan blocks in the Jordan normal form of A associated to the eigenvalue α . If α is not an eigenvalue then we set $J_\alpha(A) = 0$. Given a number α and a positive integer n we let $J(\alpha, n)$ denote the standard $n \times n$ Jordan block with eigenvalue α .

Theorem 4.5 ([18, Theorem 2], [15, Theorem 2.0.1]). *Assume that $n \leq k$. Then*

$$J(\alpha, n) \otimes J(\beta, k) = \bigoplus_{i=1}^n J(\alpha\beta, n + k + 1 - 2i).$$

In particular, $J_{\alpha\beta}(J(\alpha, n) \otimes J(\beta, k)) = \min\{n, k\}$.

We will use this to prove the following.

Lemma 4.6. *Let \mathbb{K} be an algebraically closed field and take $A \in GL_n(\mathbb{K})$ and $B \in GL_k(\mathbb{K})$. Then, for each $\lambda \in \mathbb{K}$,*

$$J_\lambda(A \otimes B) \leq \min \left\{ k \max_{\alpha \in \mathbb{K}} J_\alpha(A), n \max_{\beta \in \mathbb{K}} J_\beta(B) \right\}.$$

Proof. First, assuming that A and B have unique eigenvalues α and β respectively, the result of the lemma becomes

$$(4) \quad J_{\alpha\beta}(A \otimes B) \leq \min\{kJ_\alpha(A), nJ_\beta(B)\}.$$

We begin by proving this special case of the lemma.

Since \mathbb{K} is algebraically closed, up to conjugacy we may write A and B as the direct sum of Jordan blocks

$$A = \bigoplus_{i \in \mathbb{N}} J(\alpha, i)^{\oplus a_i}, \quad B = \bigoplus_{j \in \mathbb{N}} J(\beta, j)^{\oplus b_j}$$

where a_i, b_j count the number of Jordan blocks of size i, j in A, B respectively. Note that

$$\sum_{i \in \mathbb{N}} i a_i = n, \quad \sum_{i \in \mathbb{N}} a_i = J_\alpha(A), \quad \sum_{j \in \mathbb{N}} j b_j = k, \quad \sum_{j \in \mathbb{N}} b_j = J_\beta(B).$$

We may express $A \otimes B$ as

$$A \otimes B = \bigoplus_{i, j \in \mathbb{N}} (J(\alpha, i) \otimes J(\beta, j))^{a_i b_j}.$$

By Theorem 4.5, this leads to

$$J_{\alpha\beta}(A \otimes B) = \sum_{i, j \in \mathbb{N}} \min\{i, j\} a_i b_j \leq \sum_{i, j \in \mathbb{N}} i a_i b_j = \sum_{i \in \mathbb{N}} i a_i \sum_{j \in \mathbb{N}} b_j = n J_\beta(B).$$

A symmetric argument yields $J_{\alpha\beta}(A \otimes B) \leq k J_\alpha(A)$. This completes the proof of (4).

Now suppose the eigenvalues of A and B are not necessarily unique. Since \mathbb{K} is algebraically closed, up to conjugacy we may write A and B as direct sums

$$A = \bigoplus_{\alpha \in \mathbb{F}} A_\alpha, \quad B = \bigoplus_{\beta \in \mathbb{K}} B_\beta$$

where A_α is the direct sum of all Jordan blocks of A associated to eigenvalue α , and similarly for B_β . Suppose A_α is $n_\alpha \times n_\alpha$ and B_β is $k_\beta \times k_\beta$. Then

$$A \otimes B = \bigoplus_{\alpha, \beta \in \mathbb{K}} A_\alpha \otimes B_\beta,$$

which leads to the following, using (4):

$$J_\lambda(A \otimes B) = \sum_{\alpha\beta=\lambda} J_\lambda(A_\alpha \otimes B_\beta) \leq \sum_{\alpha\beta=\lambda} k_\beta J_\alpha(A) \leq \sum_{\beta \in \mathbb{K}} k_\beta \max_{\alpha \in \mathbb{K}} J_\alpha(A) = k \max_{\alpha \in \mathbb{K}} J_\alpha(A).$$

A symmetric argument gives the other required upper bound, proving the lemma. \square

To see how the normalized rank metric \bar{d}_{rk} behaves under tensor products, we remark that $\text{Rank}(A - \alpha \text{Id}) = n - J_\alpha(A)$ for every $\alpha \in \mathbb{K}$, implying $\bar{d}_{\text{rk}}(A, \text{Id}) = \inf_{\lambda \in \mathbb{K}} (1 - \frac{1}{n} J_\lambda(A))$. The following is thus an immediate consequence of this fact, and of Lemma 4.6.

Proposition 4.7. *Let \mathbb{F} be a field. Let $n, k \in \mathbb{N}$ and $A \in \text{GL}_n(\mathbb{F}), B \in \text{GL}_k(\mathbb{F})$. Then*

$$\bar{d}_{\text{rk}}(A \otimes B, \text{Id}) \geq \max \{ \bar{d}_{\text{rk}}(A, \text{Id}), \bar{d}_{\text{rk}}(B, \text{Id}) \}.$$

Theorem 4.8. *Let G be a linear sofic group over the field \mathbb{F} and H be a sofic group. Then $G \wr H$ is linear sofic over \mathbb{F} .*

Proof. The structure of the proof is analogous to that of Theorem 4.3. We compose the map $\Theta : G \wr H \rightarrow (\bigoplus_B \text{GL}_n(\mathbb{F})) \wr_B \text{Sym}(B)$ from Proposition 3.1 with a map Ψ giving us a map from $G \wr H$ to a linear group. We verify that $\Psi \circ \Theta$ satisfies the required almost multiplicativity and almost injectivity conditions.

Step 1: Setting the scene.

Recall that $q : \text{GL}_n(\mathbb{F}) \rightarrow \text{GL}_n(\mathbb{K})/\mathbb{K}^\times$ denotes the composition of the canonical inclusion $\text{GL}_n(\mathbb{F}) \rightarrow \text{GL}_n(\mathbb{K})$, where \mathbb{K} is the algebraic closure of \mathbb{F} , with the quotient map $\text{GL}_n(\mathbb{K}) \rightarrow \text{GL}_n(\mathbb{K})/\mathbb{K}^\times$.

Take a finite subset F of $G \wr H$ and $\varepsilon > 0$. Define $c : G \setminus \{1\} \rightarrow (0, \infty)$ to take the value $\frac{1}{16}$ for all $g \neq 1$. Let $E_G \subseteq G, E, E_H \subseteq H, c' : G \setminus \{1\} \rightarrow (0, \infty)$, and $\varepsilon' > 0$ all be as determined by F, ε , and c in Proposition 3.1. Note that from (1) in the proof of Proposition 3.1, we know that $\varepsilon' < \frac{1}{16^2} < \frac{1}{16}$. Thus, taking $\delta = \varepsilon'$ in Proposition 4.4 gives us a map $\theta : G \rightarrow \text{GL}_n(\mathbb{F})$ that is (E_G, ε') -multiplicative and is such that $q \circ \theta$ is (E_G, c) -injective.

Let $\sigma: H \rightarrow \text{Sym}(B)$, for some finite set B , be an (E_H, ε') -sofic approximation and take

$$\Theta: G \wr H \rightarrow \left(\bigoplus_B \text{GL}_n(\mathbb{F}) \right) \wr_B \text{Sym}(B)$$

to be the map constructed from θ, σ , and E in Section 3. Meanwhile, let

$$\bar{\Theta}: G \wr H \rightarrow \left(\bigoplus_B \text{GL}_n(\mathbb{K}) / \mathbb{K}^\times \right) \wr_B \text{Sym}(B)$$

be the map constructed using $q \circ \theta$ in place of θ .

We now describe how to embed the image of Θ into a linear group. First define

$$\Phi: \bigoplus_B \text{GL}_n(\mathbb{F}) \rightarrow \text{GL}((\mathbb{F}^n)^{\otimes B})$$

by

$$\Phi: (X_\beta)_{\beta \in B} \mapsto \bigotimes_{\beta \in B} X_\beta, \quad \text{for } (X_\beta)_{\beta \in B} \in \bigoplus_B \text{GL}_n(\mathbb{F}).$$

Using Φ , we define

$$\Psi: \left(\bigoplus_B \text{GL}_n(\mathbb{F}) \right) \wr_B \text{Sym}(B) \rightarrow \text{GL} \left(\bigoplus_B ((\mathbb{F}^n)^{\otimes B}) \right)$$

by

$$\Psi((A_b)_{b \in B}, \tau): (\xi_b)_{b \in B} \mapsto (\Phi(A_b) (\xi_{\tau^{-1}(b)}))_{b \in B}$$

for $(\xi_b)_{b \in B} \in \bigoplus_B ((\mathbb{F}^n)^{\otimes B})$, $(A_b)_{b \in B} \in \bigoplus_B \bigoplus_B \text{GL}_n(\mathbb{F})$, and $\tau \in \text{Sym}(B)$.

The collection of maps we have is summarized in Figure 2.

$$\begin{array}{ccccc} G \wr H & \xrightarrow{\Theta} & \left(\bigoplus_B \text{GL}_n(\mathbb{F}) \right) \wr_B \text{Sym}(B) & \xrightarrow{\Psi} & \text{GL} \left(\bigoplus_B ((\mathbb{F}^n)^{\otimes B}) \right) \\ & \searrow \bar{\Theta} & \downarrow & & \\ & & \left(\bigoplus_B \text{GL}_n(\mathbb{K}) / \mathbb{K}^\times \right) \wr_B \text{Sym}(B) & & \end{array}$$

FIGURE 2. A plan of the maps involved.

Let \tilde{d}, d_{\max} be the bi-invariant metrics on the wreath products $(\bigoplus_B \text{GL}_n(\mathbb{F})) \wr_B \text{Sym}(B)$, and $(\bigoplus_B \text{GL}_n(\mathbb{K}) / \mathbb{K}^\times) \wr_B \text{Sym}(B)$, respectively, obtained by applying Proposition 2.2 to the length functions ℓ', ℓ_{\max} on $\bigoplus_B \text{GL}_n(\mathbb{F})$, and $\bigoplus_B \text{GL}_n(\mathbb{K}) / \mathbb{K}^\times$, respectively, given by

$$\ell'(X) = d_{\text{rk}}(\Phi(X), \text{Id}),$$

$$\ell_{\max}(\bar{X}) = \max_{\beta \in B} \bar{d}_{\text{rk}}(\Phi((X_\beta)_{\beta \in B}), \text{Id}),$$

where $X = (X_\beta)_{\beta \in B} \in \bigoplus_B \text{GL}_n(\mathbb{F})$, and $\bar{X} = (q(X_\beta))_{\beta \in B} \in \bigoplus_B \text{GL}_n(\mathbb{K}) / \mathbb{K}^\times$.

Step 2: A formula for $d_{\text{rk}}(\Psi(A, \tau), \text{Id})$.

We wish to show that $\Psi \circ \Theta$ is almost multiplicative and almost injective. To do this we need a good handle on $d_{\text{rk}}(\Psi(A, \tau), \text{Id})$ when (A, τ) is in the image of Θ .

Write $A = (A_b)_{b \in B}$ with $A_b \in \bigoplus_B \text{GL}_n(\mathbb{F})$. The kernel of $\Psi(A, \tau) - \text{Id}$ is given by

$$\left\{ (\xi_b)_{b \in B} \in \bigoplus_{\substack{b \in B \\ \tau(b) \neq b}} (\mathbb{F}^n)^{\otimes B} : \Phi(A_{\tau(b)})(\xi_b) = \xi_{\tau(b)} \right\} \oplus \left(\bigoplus_{\substack{b \in B \\ \tau(b) = b}} \ker(\Phi(A_b) - \text{Id}) \right).$$

Focusing on the left term in the above direct sum, if we pick a cycle $(b_1 \ b_2 \ \dots \ b_k)$ of τ , with $k \geq 2$, then ξ_{b_1} determines ξ_{b_i} for $i = 2, \dots, k$. Thus each cycle of length greater than 1 contributes exactly $n^{|B|}$ to the dimension of the kernel. Let $\text{cyc}_0(\tau)$ be the number of cycles of length at least two in the cycle decomposition of τ . From the above discussion we see that the dimension of $\ker(\Psi(A, \tau) - \text{Id})$ is

$$n^{|B|} \text{cyc}_0(\tau) + \sum_{\substack{b \in B \\ \tau(b) = b}} \dim(\ker(\Phi(A_b) - \text{Id})).$$

It follows that

$$\begin{aligned} d_{\text{rk}}(\Psi(A, \tau), \text{Id}) &= 1 - \frac{\dim(\ker(\Psi(A, \tau) - \text{Id}))}{n^{|B|} |B|} \\ &= 1 - \frac{\text{cyc}_0(\tau)}{|B|} - \sum_{\substack{b \in B \\ \tau(b) = b}} \frac{1 - d_{\text{rk}}(\Phi(A_b), \text{Id})}{|B|}. \end{aligned}$$

Since

$$d_{\text{Hamm}}(\tau, 1) = 1 - \frac{|\{b \in B : \tau(b) = b\}|}{|B|}$$

we get

$$(5) \quad d_{\text{rk}}(\Psi(A, \tau), \text{Id}) = d_{\text{Hamm}}(\tau, 1) - \frac{\text{cyc}_0(\tau)}{|B|} + \frac{1}{|B|} \sum_{\substack{b \in B \\ \tau(b) = b}} d_{\text{rk}}(\Phi(A_b), \text{Id}).$$

Step 3: Almost multiplicativity.

Equation (5) implies that

$$d_{\text{rk}}(\Psi(A, \tau), \text{Id}) \leq \tilde{d}((A, \tau), 1).$$

Bi-invariance implies that for $(A_1, \tau_1), (A_2, \tau_2) \in (\bigoplus_B \text{GL}_n(\mathbb{F})) \wr_B \text{Sym}(B)$ we have:

$$d_{\text{rk}}(\Psi(A_1, \tau_1), (A_2, \tau_2)) \leq \tilde{d}((A_1, \tau_1), (A_2, \tau_2)).$$

Thus (F, ε) -multiplicativity of $\Psi \circ \Theta$ follows from that of Θ .

Step 4: Almost injectivity.

While for almost multiplicativity we used the almost multiplicativity of Θ , for almost injectivity we will use the almost injectivity of Θ .

Elementary calculations yield

$$d_{\text{Hamm}}(\tau, 1) = \frac{|B| - |\{b \in B : \tau(b) = b\}|}{|B|} \geq \frac{2 \text{cyc}_0(\tau)}{|B|}.$$

Using this in (5), we get that

$$d_{\text{rk}}(\Psi(A, \tau), \text{Id}) \geq \frac{1}{2} d_{\text{Hamm}}(\tau, \text{Id}) + \frac{1}{|B|} \sum_{\substack{b \in B \\ \tau(b) = b}} d_{\text{rk}}(\Phi(A_b), \text{Id}).$$

By repeated applications of Proposition 4.7 we have, for each $b \in B$,

$$d_{\text{rk}}(\Phi(A_b), \text{Id}) \geq \max_{\beta \in B} \bar{d}_{\text{rk}}(A_{b,\beta}, \text{Id}).$$

This implies that

$$d_{\text{rk}}(\Psi(A, \tau), \text{Id}) \geq \frac{1}{2} d_{\text{max}}((\bar{A}, \tau), 1).$$

where $\bar{A} = ((q(A_{b,\beta})_{\beta \in B})_{b \in B}$. If (A, τ) lies in the image of Θ then (\bar{A}, τ) lies in the image of $\bar{\Theta}$. Then, (F, c') -injectivity of $\bar{\theta}$, coupled with the above inequality, gives us $(F, \frac{c'}{2})$ -injectivity of $\Psi \circ \Theta$. \square

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